

# Dissipative Two-Fluid Models

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To Guy Boillat with friendship

## Abstract

From Hamilton's principle of stationary action, we derive governing equations of two-fluid mixtures and extend the model to the dissipative case without chemical reactions. For both conservative and dissipative cases, an algebraic identity connecting equations of momentum, mass, energy and entropy is obtained by extending the Gibbs identity in dynamics. The obtained system is hyperbolic for small relative velocity of the phases.

## 1 Introduction

The knowledge of governing equations for fluid mixtures is scientifically and industrially an important challenge. Many authors derived the governing system by using axioms of balance of mass, momentum, energy and second law of thermodynamics [1]. The mixtures were considered as a collection of different media co-existing in the physical space. For example, the balance law of momentum is given in the form :

$$\frac{\partial}{\partial t} \int_D \rho_\alpha \mathbf{u}_\alpha dD + \int_{\partial D} \rho_\alpha \mathbf{u}_\alpha u_{\alpha n} d\sigma = \int_{\partial D} T_\alpha \mathbf{n} d\sigma + \int_D \mathbf{b}_\alpha dD, \quad \alpha = 1, 2 \quad (1)$$

where  $D$  is a fixed volume,  $\partial D$  is its boundary,  $\mathbf{n}$  is the unit normal to  $\partial D$ ,  $\rho_\alpha$  are the densities of components,  $\mathbf{u}_\alpha$  are the associated velocities,  $u_{\alpha n}$  are the normal part of the velocities at  $\partial D$ ,  $T_\alpha$  are the stress tensors and  $\mathbf{b}_\alpha$  are the volume forces

associated with internal forces and interaction between components. The principle of material frame-indifference requires that  $T_\alpha$  and  $\mathbf{b}_\alpha$  depend on the thermodynamic parameters of the mixture and on the relative velocity  $\mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1$ . To include the added mass effect into consideration, one should be supposed that  $T_\alpha$  and  $\mathbf{b}_\alpha$  depend also on accelerations of phases. The structure of this dependence which is an important source of interaction, is not clear: should it be frame-indifferent or simply Galilean invariant? The method of balance laws (1) does not give a definite answer to this question.

It exists a different approach based on Hamilton's principle [2]-[4] which is used for construction of conservative (non-dissipative) mathematical models of continuous media with complex internal structure. The terms including interaction between different components of the mixtures do not require constitutive postulates difficult to interpret experimentally. They come from the direct knowledge of a unique potential for the mixture. According to [5], we call a *homogeneous mixture* if each component of the mixture occupies the whole volume of the physical space, and a *heterogeneous mixture* if each component occupies only a part of the mixture volume. In this paper, we consider only homogeneous binary mixtures, but the method can be extended to the case of heterogeneous mixtures [6]. The plan for the article is as follows :

In section 2, we formulate an extended form of Hamilton's principle of stationary action allowing us to obtain the governing equations of motion. The Lagrangian is the difference between the kinetic energy depending on the reference frame and a thermodynamic potential which is a Galilean invariant. The equations of motion introduce two new vector fields different from the velocity fields by taking into account the relative velocity of the components. They play the same role as the velocity field does in the case of a single fluid.

In section 3, the governing equations are extended to the dissipative case without chemical reactions. An algebraic identity connecting equations of momentum, equations of mass, energy equation and equations of entropy is obtained. This identity can be considered as the dynamic form of the Gibbs identity.

In section 4, we justify the compatibility of the governing equations with the second law of thermodynamics.

In section 5, Fick's law is derived. We prove that Fick's law is not yet a linear phenomenological law but a direct consequence of governing equations and Stokes drag force hypothesis.

In Section 6, we check the properties of hyperbolicity of the governing system for small relative velocity of phases.

As a convention, in the following we shall use asterisk " $\star$ " to denote *conjugate* mappings or *covectors* (vector lines); subscripts  $\alpha = 1, 2$  indicate the parameters of the  $\alpha^{th}$  component; the symbol  $I$  indicates the identity;  $\nabla$  means the gradient

operator-line;  $\nabla^*$  means the gradient operator-column;  $\mathbf{a}^* \mathbf{b}$  means the *scalar product* of vectors  $\mathbf{a}$ ,  $\mathbf{b}$  (the vector line is multiplied by the vector column);  $\mathbf{a} \mathbf{b}^*$  means the *tensor product* of vectors  $\mathbf{a}$ ,  $\mathbf{b}$  (the vector column is multiplied by the vector line);  $A \mathbf{a}$  means the product of the mapping  $A$  by a vector  $\mathbf{a}$ ;  $\mathbf{b}^* A$  means the covector  $\mathbf{c}^*$  defined by the rule  $\mathbf{c}^* = (A^* \mathbf{b})^*$ ;  $\text{div } A$  denotes the divergence of a linear transformation  $A$  which is a covector defined as follows: for any vector  $\mathbf{a}$ ,

$$\text{div } (A \mathbf{a}) = (\text{div } A) \mathbf{a} + \text{tr} \left( A \frac{\partial \mathbf{a}}{\partial \mathbf{x}} \right).$$

## 2 Governing equations in conservative case

In paper [7], we considered a pure *mechanical* case (without entropy). Now we consider the general case. We take the Lagrangian of the binary system in the following form:

$$L = \sum_{\alpha=1}^2 \frac{1}{2} \rho_{\alpha} \mathbf{u}_{\alpha}^2 - \rho_{\alpha} \Omega_{\alpha} - W(\rho_1, \rho_2, s_1, s_2, \mathbf{w}) \quad (2)$$

where in the whole paper the summation is over fluid components ( $\alpha = 1, 2$ ) and  $\rho_{\alpha}$  are the densities of components,  $s_{\alpha}$  are the specific entropies,  $\mathbf{u}_{\alpha}$  are the velocities,  $\mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1$  is the relative velocity,  $\Omega_{\alpha}$  are the external force potentials,  $W$  is a potential per unit volume of the mixture. The dependance of  $W$  with respect to the relative velocity is analog to take into account the added mass effect in heterogeneous two-fluid theory as it was done by Geurst [3]. The fact that  $W$  depends on two entropies is classically adopted in the literature [2, 4, 8, 9]. The potential  $W$  is related with the internal energy  $U$  of the mixture through the transformation

$$U = W - \frac{\partial W}{\partial \mathbf{w}} \mathbf{w} \quad (3)$$

so that the total energy of the system is [7]:

$$\varepsilon = \sum_{\alpha=1}^2 \frac{1}{2} \rho_{\alpha} \mathbf{u}_{\alpha}^2 + \rho_{\alpha} \Omega_{\alpha} + U$$

Let us note that to define the internal energy of one-velocity media it is useful to consider a moving coordinate system in which the elementary volume of the continuum is at rest. The total energy of the continuum with respect to this system is called the internal energy of the medium. For a two-velocity medium, there is no reference frame in which any motion could be disregarded. This is the reason why the standard definition of internal energy is dependent on the relative motion of components.

The formula (3) implies that the internal energy  $U$  is a Galilean invariant. The dependence of  $U$  (or  $W$ ) on  $\mathbf{w}$  is an important property of multicomponent fluid mixtures.

Let  $\mathbf{x}$  be the Eulerian coordinates,  $t$  be the time,  $\mathbf{X}_\alpha$  be the Lagrangian coordinates of each component. The mass and the entropy conservation laws in the Eulerian coordinates are:

$$\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div} (\rho_\alpha \mathbf{u}_\alpha) = 0, \quad \frac{\partial}{\partial t} (\rho_\alpha s_\alpha) + \operatorname{div} (\rho_\alpha s_\alpha \mathbf{u}_\alpha) = 0. \quad (4)$$

In the Lagrangian coordinates, equations (4) are equivalent to:

$$\rho_\alpha \det F_\alpha = \rho_{\alpha 0} (\mathbf{X}_\alpha), \quad s_\alpha = s_{\alpha 0} (\mathbf{X}_\alpha), \quad (5)$$

where

$$F_\alpha = \frac{\partial \mathbf{x}}{\partial \mathbf{X}_\alpha} \quad (6)$$

is the deformation gradient at  $\mathbf{X}_\alpha$ ;  $\rho_{\alpha 0} (\mathbf{X}_\alpha)$  and  $s_{\alpha 0} (\mathbf{X}_\alpha)$  do not depend on  $t$ . The relation between the Eulerian and Lagrangian coordinates is given by the local diffeomorphism  $\mathbf{x} = \phi_\alpha (\mathbf{X}_\alpha, t)$ , where  $\phi_\alpha (\mathbf{X}_\alpha, t)$  is the solution of the Cauchy problem :

$$\frac{d\phi_\alpha}{dt} = \mathbf{u} (\phi_\alpha, t), \quad \phi_\alpha (\mathbf{X}_\alpha, 0) = \mathbf{X}_\alpha$$

Let  $\mathbf{X}_\alpha = \psi_\alpha (\mathbf{x}, t)$  be its inverse mapping ( $\phi_\alpha \circ \psi_\alpha = I$ ). We define the virtual motion of the mixture such that [4, 10]:

$$\mathbf{x} = \Phi_\alpha (\mathbf{X}_\alpha, t, \varepsilon_\alpha), \quad \mathbf{X}_\alpha = \Psi_\alpha (\mathbf{x}, t, \varepsilon_\alpha), \quad \Phi_\alpha \circ \Psi_\alpha = I$$

$$\Phi_\alpha (\mathbf{X}_\alpha, t, 0) = \phi_\alpha (\mathbf{X}_\alpha, t), \quad \Psi_\alpha (\mathbf{x}, t, 0) = \psi_\alpha (\mathbf{x}, t),$$

where  $\varepsilon_\alpha$  belong to a vicinity of zero. The Lagrangian and Eulerian virtual displacements are defined respectively as :

$$\delta \mathbf{X}_\alpha = \frac{\partial \Psi_\alpha}{\partial \varepsilon_\alpha} (\mathbf{x}, t, \varepsilon_\alpha) |_{\varepsilon_\alpha=0}, \quad \delta_\alpha \mathbf{x} = \frac{\partial \Phi_\alpha}{\partial \varepsilon_\alpha} (\mathbf{X}_\alpha, t, \varepsilon_\alpha) |_{\varepsilon_\alpha=0}. \quad (7)$$

The definitions (6-7) imply the following relation between  $\delta \mathbf{X}_\alpha$  and  $\delta_\alpha \mathbf{x}$  [4]:

$$\delta_\alpha \mathbf{x} = -F_\alpha \delta \mathbf{X}_\alpha \quad (8)$$

The variations of  $\mathbf{u}_\alpha (t, \mathbf{x})$ ,  $\rho_\alpha (t, \mathbf{x})$  and  $s_\alpha (t, \mathbf{x})$  are deduced from (6) - (7) and from the definition of the Lagrangian coordinates  $\mathbf{X}_\alpha$  :

$$\frac{d_\alpha \mathbf{X}_\alpha}{dt} = 0, \quad \frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha^* \nabla^*.$$

We obtain in Appendix A the values of  $\delta \mathbf{u}_\alpha(\mathbf{x}, t)$ ,  $\delta \rho_\alpha(\mathbf{x}, t)$  and  $\delta s_\alpha(\mathbf{x}, t)$ , where  $\delta f(t, \mathbf{x})$  means the variation of  $f$  when  $t, \mathbf{x}$  are fixed and  $\text{div}_\alpha(\delta \mathbf{X}_\alpha)$  means the divergence with respect to the coordinates  $\mathbf{X}_\alpha$ . We note that in [7] we used different but equivalent expressions for these variations. Using the definition (2) of the Lagrangian  $L$  as a function of  $\rho_\alpha, \mathbf{u}_\alpha, s_\alpha$ , we introduce the following quantities:

$$\begin{cases} R_\alpha \equiv \frac{\partial L}{\partial \rho_\alpha} = \frac{1}{2} \mathbf{u}_\alpha^2 - \frac{\partial W}{\partial \rho_\alpha} - \Omega_\alpha, \\ \mathbf{K}_\alpha^\star \equiv \frac{1}{\rho_\alpha} \frac{\partial L}{\partial \mathbf{u}_\alpha} = \mathbf{u}_\alpha^\star - \frac{(-1)^\alpha}{\rho_\alpha} \frac{\partial W}{\partial \mathbf{w}}, \\ \rho_\alpha \theta_\alpha \equiv -\frac{\partial L}{\partial s_\alpha} = \frac{\partial W}{\partial s_\alpha} \end{cases} \quad (9)$$

We note that a best set of independent variables is :  $\rho_\alpha, \mathbf{j}_\alpha = \rho_\alpha \mathbf{u}_\alpha, \rho_\alpha s_\alpha$ . However, in this case, the corresponding derivation should be given in four-dimensional space [11]. For the sake of simplicity we use the first set of independent variables. The last formula defines the thermodynamic temperature  $\theta_\alpha$  of each component which is now a dynamical quantity depending on the relative velocity of the components.

Let  $\omega = D \times [t_1, t_2]$  be the domain in the four-dimensional space  $(\mathbf{x}, t)$  and  $\omega_\alpha$  be its image in the  $(\mathbf{X}_\alpha, t)$ -space. Here  $[t_1, t_2]$  is a time interval and  $D$  is a fixed domain. We consider Hamilton's principle in the form,

$$\delta_\alpha a \equiv \delta_\alpha \int_\omega L d\omega = 0$$

under constraints (4) where  $\delta_\alpha a$  are variations of  $a$  associated with the variation of  $\mathbf{X}_\alpha = \Psi_\alpha(\mathbf{x}, t, \varepsilon_\alpha)$ . It means that  $\delta_\alpha a = \frac{da}{d\varepsilon_\alpha} |_{\varepsilon_\alpha} = 0$ . We have to emphasis on the fact that the domain  $D$  is fixed in the physical space. This particularity is related to the impossibility to have material volume in general motion of the mixture. Taking into account formulae (8), variations in Appendix A and definitions (9), we get

$$\begin{aligned} \delta_\alpha a &= \int_\omega \left( R_\alpha \delta \rho_\alpha + \rho_\alpha \mathbf{K}_\alpha^\star \delta \mathbf{u}_\alpha - \rho_\alpha \theta_\alpha \delta s_\alpha \right) d\omega \\ &= \int_{\omega_\alpha} \left( R_\alpha \text{div}_\alpha(\rho_{\alpha 0} \delta \mathbf{X}_\alpha) - \rho_{\alpha 0} \mathbf{K}_\alpha^\star F_\alpha \frac{\partial}{\partial t}(\delta \mathbf{X}_\alpha) - \rho_{\alpha 0} \theta_\alpha \frac{\partial s_{\alpha 0}}{\partial \mathbf{X}_\alpha} \delta \mathbf{X}_\alpha \right) d\omega_\alpha. \end{aligned}$$

In the last expression all quantities are considered as functions of  $(\mathbf{X}_\alpha, t)$ . Hence,

$$\delta_\alpha a = \int_{\omega_\alpha} \rho_{\alpha 0} \left( -\frac{\partial R_\alpha}{\partial \mathbf{X}_\alpha} + \frac{\partial}{\partial t}(\mathbf{K}_\alpha^\star F_\alpha) - \theta_\alpha \frac{\partial s_{\alpha 0}}{\partial \mathbf{X}_\alpha} \right) \delta \mathbf{X}_\alpha d\omega_\alpha$$

$$+ \int_{\omega_\alpha} \text{Div}_\alpha(\rho_{\alpha 0} \mathbf{G}) d\omega_\alpha = 0$$

where  $\mathbf{G} = (R_\alpha \delta \mathbf{X}_\alpha, -\mathbf{K}_\alpha^* F_\alpha \delta \mathbf{X}_\alpha)$  and  $\text{Div}_\alpha$  is the divergence operator in the 4-dimensional space  $\omega_\alpha$  associated with  $(\mathbf{X}_\alpha, t)$ . All the functions are assumed to be smooth enough in the domain  $\omega_\alpha$  and  $\delta \mathbf{X}_\alpha = 0$  on  $\partial \omega_\alpha$ . Then, we obtain the equations of motion for each component in Lagrangian coordinates:

$$\frac{\partial}{\partial t} (\mathbf{K}_\alpha^* F_\alpha) - \frac{\partial R_\alpha}{\partial \mathbf{X}_\alpha} - \theta_\alpha \frac{\partial s_{\alpha 0}}{\partial \mathbf{X}_\alpha} = 0 \quad (10)$$

Taking into account the identity  $\frac{d_\alpha F_\alpha}{dt} - \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} F_\alpha = 0$ , we rewrite (10) in Eulerian coordinates in the form :

$$\frac{d_\alpha \mathbf{K}_\alpha^*}{dt} + \mathbf{K}_\alpha^* \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} = \frac{\partial R_\alpha}{\partial \mathbf{x}} + \theta_\alpha \frac{\partial s_\alpha}{\partial \mathbf{x}} \quad (11)$$

If  $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$ , then  $\mathbf{K}_\alpha = \mathbf{u}$  and (10) is equivalent to

$$\frac{d\mathbf{u}}{dt} + \nabla^*(h + \Omega) = \theta \nabla^* s, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}^* \nabla^*$$

where  $h$  is the enthalpy and  $\Omega$  is an external potential [10]. Conservations of the total momentum and the total energy are a consequence of the governing equations (4), (11):

$$\sum_{\alpha=1}^2 \frac{\partial \rho_\alpha \mathbf{K}_\alpha^*}{\partial t} + \text{div} \left( \rho_\alpha \mathbf{u}_\alpha \mathbf{K}_\alpha^* + \left( \rho_\alpha \frac{\partial W}{\partial \rho_\alpha} - W \right) I \right) + \rho_\alpha \frac{\partial \Omega_\alpha}{\partial \mathbf{x}} = 0 \quad (12)$$

$$\sum_{\alpha=1}^2 \frac{\partial}{\partial t} \left( \rho_\alpha \left( \frac{1}{2} \mathbf{u}_\alpha^2 + \Omega_\alpha \right) + U \right) + \text{div} \left( \rho_\alpha \mathbf{u}_\alpha (\mathbf{K}_\alpha^* \mathbf{u}_\alpha - R_\alpha) \right) - \rho_\alpha \frac{\partial \Omega_\alpha}{\partial t} = 0 \quad (13)$$

The covector  $\mathbf{K}_\alpha^*$  is an essential quantity; indeed,  $\rho_\alpha \mathbf{K}_\alpha^*$  (but not  $\rho_\alpha \mathbf{u}_\alpha$ ) is the impulse for the  $\alpha^{th}$  component of the mixture. Moreover, for adiabatic motions, the definition of potential flows for two-component mixtures is associated with  $\text{rot } \mathbf{K}_\alpha = 0$  and not with  $\text{rot } \mathbf{u}_\alpha = 0$ . For potential motion, equation (11) yields additional conservation laws [7, 11]:

$$\frac{\partial \mathbf{K}_\alpha^*}{\partial t} + \nabla (\mathbf{K}_\alpha^* \mathbf{u}_\alpha - R_\alpha) = 0$$

In the particular case of bubbly liquids, Geurst was the first to carry out a term analog to  $\mathbf{K}_\alpha$  [3]. Moreover, if  $\text{rot } \mathbf{K}_\alpha \neq 0$ , the system of governing equations is not conservative in terms of  $\mathbf{K}_\alpha$ ,  $\rho_\alpha$  and  $s_\alpha$  (the number of conservation laws admitted by the system is less than the number of unknown variables); but, nevertheless the system can be rewritten in conservative form if we add the gradient tensor  $F_\alpha$  as unknown variable [7].

### 3 Governing equations in the dissipative case and dynamic Gibbs identity

The conservative fluid mixture model presented in section 2 is relevant to the *first gradient theory* [12]: the forces applied to the continuous medium are divided into volume forces and surface forces. In fluid mixture flows it is reasonable to neglect the surface friction forces compared to galilean invariant algebraic volume forces. The virtual work  $\delta\mathcal{T}_\alpha$  of dissipative forces applied to the  $\alpha^{th}$  component is in the form  $\delta\mathcal{T}_\alpha = \mathbf{f}_\alpha^* \delta_\alpha \mathbf{x}$ . For the same virtual displacement of two components,

$\delta \mathbf{x} = \delta_\alpha \mathbf{x}_1 = \delta_\alpha \mathbf{x}_2$ , the total virtual work of dissipative forces is  $\delta\mathcal{T} = \sum_{\alpha=1}^2 \mathbf{f}_\alpha^* \delta \mathbf{x}$ .

For a solid displacement, the work  $\delta\mathcal{T}$  is equal to zero and consequently  $\sum_{\alpha=1}^2 \mathbf{f}_\alpha^* = 0$ .

We specify later the behavior of forces  $\mathbf{f}_\alpha^*$ . Let us introduce the quantities  $\mathbf{M}_\alpha$ ,  $B_\alpha$ ,  $S$  and  $E$  such that:

$$\begin{aligned} \mathbf{M}_\alpha^* &= \rho_\alpha \frac{d_\alpha \mathbf{K}_\alpha^*}{dt} + \rho_\alpha \mathbf{K}_\alpha^* \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} - \rho_\alpha \frac{\partial R_\alpha}{\partial \mathbf{x}} - \rho_\alpha \theta_\alpha \frac{\partial s_\alpha}{\partial \mathbf{x}} - \mathbf{f}_\alpha^* \\ B_\alpha &= \frac{\partial \rho_\alpha}{\partial t} + \text{div}(\rho_\alpha \mathbf{u}_\alpha) \\ S &= \sum_{\alpha=1}^2 \rho_\alpha \theta_\alpha \frac{d_\alpha s_\alpha}{dt} + \mathbf{f}_\alpha^* \mathbf{u}_\alpha \\ E &= \sum_{\alpha=1}^2 \frac{\partial}{\partial t} \left( \rho_\alpha \left( \frac{1}{2} \mathbf{u}_\alpha^2 + \Omega_\alpha \right) + U \right) + \text{div} \left( \rho_\alpha \mathbf{u}_\alpha (\mathbf{K}_\alpha^* \mathbf{u}_\alpha - R_\alpha) \right) - \rho_\alpha \frac{\partial \Omega_\alpha}{\partial t} \end{aligned}$$

We prove in Appendix B the following property:

**Theorem:** *For any motion of the mixture, we have the identity*

$$E - \sum_{\alpha=1}^2 \left( \mathbf{M}_\alpha^* \mathbf{u}_\alpha + (\mathbf{K}_\alpha^* \mathbf{u}_\alpha - R_\alpha) B_\alpha \right) - S \equiv 0$$

This relation is the most general expression of the *Gibbs identity* in dynamics. Analogous identities were obtained earlier for thermocapillary mixtures [4] and bubbly liquids [13]. For each component of the mixture, equation of momentum and equation of mass are in the form

$$\mathbf{M}_\alpha^* = 0, B_\alpha = 0 \quad (14)$$

The Gibbs identity implies  $S = E$ . Hence, the equation of the entropy  $S = 0$ ,

$$\sum_{\alpha=1}^2 \rho_{\alpha} \theta_{\alpha} \frac{d_{\alpha} s_{\alpha}}{dt} + \mathbf{f}_{\alpha}^* \mathbf{u}_{\alpha} = 0 \quad (15)$$

is equivalent to the equation of the energy  $E = 0$ . We note also that the equations  $\mathbf{M}_{\alpha}^* = 0$  and  $B_{\alpha} = 0$  imply conservation of the total momentum of the mixture  $\mathbf{M} = 0$  (see (12))

## 4 The second law of thermodynamics

In conservative case, the system with two different entropies is closed by (4). In dissipative case we need additional arguments to obtain equations for each entropy  $s_{\alpha}$  which could replace equations (4). We take these equations in the form :

$$\rho_{\alpha} \theta_{\alpha} \frac{d_{\alpha} s_{\alpha}}{dt} + \mathbf{f}_{\alpha}^* (\mathbf{u}_{\alpha} - \mathbf{u}) + q_{\alpha} = 0 \quad (16)$$

which must be compatible with (15). Here  $\rho \mathbf{u} = \sum_{\alpha=1}^2 \rho_{\alpha} \mathbf{u}_{\alpha}$  is the total momentum,

$\rho \sum_{\alpha=1}^2 \rho_{\alpha}$  and  $\sum_{\alpha=1}^2 q_{\alpha} = 0$ . The last relation means that we have only internal heat exchanges between components. Consequently, if

$$\sum_{\alpha=1}^2 \frac{\mathbf{f}_{\alpha}^*}{\theta_{\alpha}} (\mathbf{u}_{\alpha} - \mathbf{u}) + \frac{q_{\alpha}}{\theta_{\alpha}} \leq 0, \quad (17)$$

we obtain the entropy inequality [8, 9]

$$\sum_{\alpha=1}^2 \rho_{\alpha} \frac{d_{\alpha} s_{\alpha}}{dt} \geq 0$$

Let us note that relation (17) is verified if

$$\mathbf{f}_1 = k \left( \frac{\mathbf{u}_2 - \mathbf{u}}{\theta_2} - \frac{\mathbf{u}_1 - \mathbf{u}}{\theta_1} \right), \quad \mathbf{f}_2 = \mathbf{f}_1, \quad k > 0$$

and

$$q_1 = \kappa \left( \frac{1}{\theta_2} - \frac{1}{\theta_1} \right), \quad q_1 = -q_2, \quad \kappa > 0$$

The fact that the inverse temperatures (coldness) appear in the closure relations play an important role in other applications [14].



## 5 Fick's law as a consequence of the governing equations

The governing equations for each component are :

$$\mathbf{M}_\alpha^\star \equiv \rho_\alpha \frac{d_\alpha \mathbf{K}_\alpha^\star}{dt} + \rho_\alpha \mathbf{K}_\alpha^\star \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} - \rho_\alpha \frac{\partial R_\alpha}{\partial \mathbf{x}_\alpha} - \rho_\alpha \theta_\alpha \frac{\partial s_\alpha}{\partial \mathbf{x}} - \mathbf{f}_\alpha^\star = 0$$

For slow isothermal motions ( $\theta_1 = \theta_2 = \theta_0 = \text{const}$ ), we can rewrite these equations in the following approximate form:

$$\mathbf{M}_\alpha^\star \simeq \rho_\alpha \frac{\partial}{\partial \mathbf{x}} \frac{\partial W}{\partial \rho_\alpha} - \rho_\alpha \theta_0 \frac{\partial s_\alpha}{\partial \mathbf{x}} - \mathbf{f}_\alpha^\star = 0$$

or

$$\mathbf{M}_\alpha^\star \simeq \rho_\alpha \frac{\partial \mu_\alpha}{\partial \mathbf{x}} - \mathbf{f}_\alpha^\star = 0$$

where  $\mu_\alpha = \frac{\partial W}{\partial \rho_\alpha} - \theta_0 s_\alpha$  is the chemical potential for the  $\alpha^{th}$  phase. Considering the difference  $\mathbf{M}_2^\star - \mathbf{M}_1^\star$  we obtain :

$$\nabla \mu = \frac{\mathbf{f}_2^\star}{\rho_2} - \frac{\mathbf{f}_1^\star}{\rho_1} \equiv \frac{\rho \mathbf{f}^\star}{\rho_1 \rho_2} \quad (18)$$

where  $\mathbf{f}^\star = -\mathbf{f}_1^\star$  and  $\mu = \mu_2 - \mu_1$ . Equation (18) is the general form of Fick's law. So, Fick's law is not a linear phenomenological law but a direct consequence of equations of motion and the Stokes drag hypothesis which was previously noticed by Bowen with an other model [1].

## 6 Hyperbolicity of the two-fluid mixture model

The hyperbolicity of governing equations is very important because it implies well-posedness of the Cauchy problem. In mechanical case (when we neglect the equations of entropies) we are back to our previous study [7, 11]. The only difference is the right-hand side algebraic terms  $\mathbf{f}_\alpha$  due to the Stokes-like drag forces. Obviously, they do not affect the hyperbolicity analysis. The potential  $W$  is then a function of  $\rho_1, \rho_2$  and  $w = |\mathbf{w}|$ . The Lagrangian is (with  $\Omega_\alpha = 0$ )

$$L = \sum_{\alpha=1}^2 \frac{1}{2} \rho_\alpha \mathbf{u}_\alpha^2 - W(\rho_1, \rho_2, w)$$

We gave a sufficient condition of the hyperbolicity of system (4), (12) in the multi-dimensional irrotational case where  $\text{rot } \mathbf{K}_\alpha = 0$ . Recall the main results we

obtained in this case: after a change of variables, system ((4),(12)) takes the form

$$\frac{\partial}{\partial t} \left( \frac{\partial G}{\partial \sigma_\alpha} \right) - \operatorname{div} \left( \frac{\partial}{\partial \sigma_\alpha} \left( \sum_{\beta=1}^2 \sigma_\beta \mathbf{j}_\beta \right) \right) = 0, \quad (19)$$

$$\frac{\partial}{\partial t} \left( \frac{\partial G}{\partial \mathbf{j}_\alpha} \right) - \operatorname{div} \left( \frac{\partial}{\partial \mathbf{j}_\alpha} \left( \sum_{\beta=1}^2 \sigma_\beta \mathbf{j}_\beta \right) \right) = 0. \quad (20)$$

where

$$G(\sigma_1, \sigma_2, \mathbf{j}_1, \mathbf{j}_2) = L(\rho_1, \rho_2, \mathbf{j}_1, \mathbf{j}_2) - \sum_{\alpha=1}^2 \sigma_\alpha \rho_\alpha, \quad \text{with } \sigma_\alpha = \frac{\partial L}{\partial \rho_\alpha}$$

The function  $G$  is a partial Legendre transformation of  $L(\rho_1, \rho_2, \mathbf{j}_1, \mathbf{j}_2)$  with respect to the variables  $\rho_\alpha$ :

$$\frac{\partial G}{\partial \sigma_\alpha} = -\rho_\alpha, \quad \frac{\partial G}{\partial \mathbf{j}_\alpha} = \mathbf{K}_\alpha^*,$$

and the system (19), (20) can be rewritten in a symmetric form [15, 16, 17]

$$A \frac{\partial \mathbf{u}}{\partial t} + B^i \frac{\partial \mathbf{u}}{\partial x^i} = 0, \quad A = A^*, \quad B^i = (B^i)^*, \quad i = 1, 2, 3 \quad (21)$$

where

$$\mathbf{u}^* = (\sigma_1, \sigma_2, \mathbf{j}_1^*, \mathbf{j}_2^*), \quad A = \frac{\partial^2 G}{\partial \mathbf{u}^2}$$

and the matrices  $B^i$  can be obtained from (19), (20). If  $A$  is positive definite, system (21) is hyperbolic. We proved in [7, 11] that the conditions

$$\frac{\partial^2 W}{\partial w^2} < 0, \quad \frac{\partial^2 W}{\partial \rho_1^2} > 0, \quad \frac{\partial^2 W}{\partial \rho_1^2} \frac{\partial^2 W}{\partial \rho_2^2} - \left( \frac{\partial^2 W}{\partial \rho_1 \partial \rho_2} \right)^2 > 0 \quad (22)$$

guarantee the hyperbolicity of our system for small relative velocity of phases. Due to relation (3), the inequalities (22) mean the convexity of the internal energy  $U$  that corresponds to a natural condition of stability. Finally, we established that the stability implies the hyperbolicity of the governing equations for small relative velocity  $\mathbf{w}$ , provided that  $\operatorname{rot} \mathbf{K}_\alpha = 0$  (condition always fulfilled for one-dimensional flows).

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### Appendix A.

The definition of the Lagrangian coordinates  $\mathbf{X}_\alpha$  implies  $\frac{\partial \mathbf{X}_\alpha}{\partial t} + \frac{\partial \mathbf{X}_\alpha}{\partial \mathbf{x}} \mathbf{u}_\alpha = 0$ . Taking the derivative with respect to  $\varepsilon_\alpha$  at zero, we obtain the following equation :

$$\frac{\partial \delta \mathbf{X}_\alpha}{\partial t} + \frac{\partial \delta \mathbf{X}_\alpha}{\partial \mathbf{x}} \mathbf{u}_\alpha + \frac{\partial \mathbf{X}_\alpha}{\partial \mathbf{x}} \delta \mathbf{u}_\alpha = 0, \quad \delta \mathbf{u}_\alpha(\mathbf{x}, t) = -F_\alpha \frac{d_\alpha}{dt} (\delta \mathbf{X}_\alpha)$$

Equation (5) yields :

$$\delta \rho_\alpha(\mathbf{x}, t) \det F_\alpha(\mathbf{x}, t) + \rho_\alpha \delta(\det F_\alpha) = \frac{\partial \rho_{\alpha 0}}{\partial \mathbf{X}_\alpha} \delta \mathbf{X}_\alpha \quad (23)$$

Using the Euler-Jacobi identity,  $\delta(\det F_\alpha) = \det F_\alpha(\mathbf{x}, t) \operatorname{tr} \left( F_\alpha^{-1} \delta F_\alpha \right)$  and  $\delta F_\alpha(\mathbf{x}, t) = -F_\alpha(\mathbf{x}, t) \delta F^{-1}(\mathbf{x}, t) F_\alpha(\mathbf{x}, t)$ ,  $\delta F^{-1}(\mathbf{x}, t) = \frac{\partial \delta \mathbf{X}_\alpha}{\partial \mathbf{x}}$ , we deduce :

$$\delta(\det F_\alpha)(\mathbf{x}, t) = -\det F_\alpha \operatorname{tr} \left( \delta F^{-1} F_\alpha \right) = -\det F_\alpha \operatorname{tr} \left( \frac{\partial \delta \mathbf{X}_\alpha}{\partial \mathbf{X}_\alpha} \right). \text{ Or,}$$

$$\delta \det F_\alpha = -\det F_\alpha \operatorname{div}_\alpha(\delta \mathbf{X}_\alpha) \quad (24)$$

Substituting (24) into (23) we obtain :

$$\delta \rho_\alpha(\mathbf{x}, t) = \rho_\alpha \operatorname{div}_\alpha(\delta \mathbf{X}_\alpha) + \frac{\rho_\alpha}{\rho_{\alpha 0}} \frac{\partial \rho_{\alpha 0}}{\partial \mathbf{X}_\alpha} \delta \mathbf{X}_\alpha = \frac{\operatorname{div}_\alpha(\rho_{\alpha 0} \delta \mathbf{X}_\alpha)}{\det F_\alpha}$$

$$\delta s_\alpha(\mathbf{x}, t) = \frac{\partial s_{\alpha 0}}{\partial \mathbf{X}_\alpha} \delta \mathbf{X}_\alpha$$

## Appendix B.

The proof of the Gibbs identity is obtained by summing the following algebraic identities  $a - f$  :

For dissipative terms,

**a.**

$$\mathbf{f}_1^* \mathbf{u}_1 + \mathbf{f}_2^* \mathbf{u}_2 - \mathbf{f}_1^* \mathbf{u}_1 - \mathbf{f}_2^* \mathbf{u}_2 \equiv 0$$

For the external potentials  $\Omega_\alpha$ ,

**b.**

$$\frac{\partial}{\partial t} \rho_\alpha \Omega_\alpha + \text{div} (\rho_\alpha \Omega_\alpha \mathbf{u}_\alpha) - \rho_\alpha \frac{\partial \Omega_\alpha}{\partial \mathbf{x}} \mathbf{u}_\alpha - B_\alpha \Omega_\alpha - \rho_\alpha \frac{\partial \Omega_\alpha}{\partial t} \equiv 0$$

For the velocity fields  $\mathbf{u}_\alpha$ ,

**c.**

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{1}{2} \rho_\alpha \mathbf{u}_\alpha^2 \right) + \text{div} \left( \rho_\alpha \mathbf{u}_\alpha (\mathbf{u}_\alpha^2 - \frac{1}{2} \mathbf{u}_\alpha^2) \right) \\ & - B_\alpha \left( \mathbf{u}_\alpha^2 - \frac{1}{2} \mathbf{u}_\alpha^2 \right) - \left( \rho_\alpha \frac{d_\alpha \mathbf{u}_\alpha^*}{dt} + \rho_\alpha \mathbf{u}_\alpha^* \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} - \rho_\alpha \frac{\partial}{\partial \mathbf{x}} \left( \frac{1}{2} \mathbf{u}_\alpha^2 \right) \right) \mathbf{u}_\alpha \equiv 0 \end{aligned}$$

Let us introduce  $\mathbf{i}^* = - \frac{\partial W}{\partial \mathbf{w}}$ . Then the expression,

$$\frac{\partial U}{\partial t} \equiv \frac{\partial}{\partial t} \left( W - \frac{\partial W}{\partial \mathbf{w}} \mathbf{w} \right) = \frac{\partial \mathbf{i}^*}{\partial t} \mathbf{w} + \sum_{\alpha=1}^2 \left( \frac{\partial W}{\partial \rho_\alpha} \frac{\partial \rho_\alpha}{\partial t} + \rho_\alpha \theta_\alpha \frac{\partial s_\alpha}{\partial t} \right)$$

and the three following identities  $d - f$  prove the formula.

**d.**

$$\sum_{\alpha=1}^2 \frac{\partial W}{\partial \rho_\alpha} \frac{\partial \rho_\alpha}{\partial t} + \text{div} \left( \frac{\partial W}{\partial \rho_\alpha} \rho_\alpha \mathbf{u}_\alpha \right) - \rho_\alpha \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial W}{\partial \rho_\alpha} \right) \mathbf{u}_\alpha - \frac{\partial W}{\partial \rho_\alpha} \left( \frac{\partial \rho_\alpha}{\partial t} + \text{div} (\rho_\alpha \mathbf{u}_\alpha) \right) \equiv 0$$

**e.**

$$\sum_{\alpha=1}^2 \rho_\alpha \theta_\alpha \frac{\partial s_\alpha}{\partial t} + \rho_\alpha \theta_\alpha \frac{\partial s_\alpha}{\partial \mathbf{x}} \mathbf{u}_\alpha - \rho_\alpha \theta_\alpha \frac{d_\alpha s_\alpha}{dt} \equiv 0$$

**f.**

$$\begin{aligned} & \frac{\partial \mathbf{i}^*}{\partial t} \mathbf{w} + \sum_{\alpha=1}^2 \text{div} \left( (-1)^\alpha \left( \frac{\mathbf{i}^*}{\rho_\alpha} \mathbf{u}_\alpha \right) \rho_\alpha \mathbf{u}_\alpha \right) - \left( \rho_\alpha \frac{d_\alpha}{dt} \left( (-1)^\alpha \frac{\mathbf{i}^*}{\rho_\alpha} \right) + \rho_\alpha (-1)^\alpha \frac{\mathbf{i}^*}{\rho_\alpha} \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} \right) \mathbf{u}_\alpha \\ & - (-1)^\alpha \left( \frac{\mathbf{i}^*}{\rho_\alpha} \mathbf{u}_\alpha \right) \left( \frac{\partial \rho_\alpha}{\partial t} + \text{div} (\rho_\alpha \mathbf{u}_\alpha) \right) \equiv 0 \end{aligned}$$